

Strong Asymptotics of the Generating Polynomials of the Stirling Numbers of the Second Kind

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For the horizontal generating functions $P_n(z) = \sum_{k=1}^n S(n, k) z^k$ of the Stirling numbers of the second kind, strong asymptotics are established, as $n \rightarrow \infty$. By using the saddle point method for $Q_n(z) = P_n(nz)$ there are two main results: an oscillating asymptotic for $z \in (-e, 0)$ and a uniform asymptotic on every compact subset of $\mathbb{C} \setminus [-e, 0]$. Finally, an Airy asymptotic in the neighborhood of $-e$ is deduced.

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1. INTRODUCTION AND SUMMARY

This paper contains asymptotic expansions for the horizontal generating function of the Stirling numbers of the second kind $S(n, k)$, which are defined by the following double generating function (see [3, p. 50]):

$$\exp\{z(e^u - 1)\} =: 1 + \sum_{1 \leq k \leq n < \infty} S(n, k) \frac{u^n}{n!} z^k, \quad z, u \in \mathbb{C}. \quad (1.1)$$

The horizontal generating functions $P_n(z)$ are the coefficients of the power series

$$\exp\{z(e^u - 1)\} =: 1 + \sum_{n=1}^{\infty} \frac{P_n(z)}{n!} u^n, \quad z, u \in \mathbb{C},$$

that gives

$$P_n(z) = \sum_{k=1}^n S(n, k) z^k = \frac{n!}{2\pi i} \int_{\gamma_0} \frac{\exp\{z(e^t - 1)\}}{t^{n+1}} dt, \quad (1.2)$$

with γ_0 a simple closed curve with positive orientation encircling 0. In contrast to the vertical generating function and to the corresponding functions of the Stirling numbers of the first kind, whose particular sums can be computed exactly (see [3, pp. 206, 212]), there is no comparative result for the function P_n . Therefore, it is interesting to at least deduce some asymptotic results. Concerning asymptotic characteristics only the case $P_n(1)$, the so-called Bell number, has been investigated so far (see [1], [2, pp. 102–108], [3, pp. 296–297], [6]). Moreover, there is one result respecting the zeros of P_n . These are simple, real, and not greater than 0 (see [3, p. 271]). In this work, we deduce two asymptotic expansions for P_n with the help of the saddle point method, which requires that the saddle point and the parameter n are independent of each other. This leads to the function

$$Q_n(z) := P_n(nz) \\ = \frac{n!}{2\pi i} \int_{\gamma_0} e^{-n(\ln t - z(e^t - 1))} \frac{dt}{t}, \quad n \in \mathbb{N}, \quad z \in \mathbb{C}. \quad (1.3)$$

In accordance with asymptotic results for the classic orthogonal polynomials, see for example the Hermite polynomials [8, p. 201], we obtain the following asymptotics of the Plancherel–Rotach-type:

(i) With $\phi \in (0, \pi)$ there is the oscillating asymptotics

$$Q_n\left(-\frac{\sin \phi}{\phi} e^{\phi \cot \phi}\right) = k_n(\phi) \left(\sin\left(n\left(\pi - \phi + \frac{\sin^2 \phi}{\phi}\right) + \eta(\phi)\right) + \mathcal{O}\left(\frac{1}{n}\right) \right),$$

with $k_n(\phi) > 0$ and $\eta(\phi)$ bounded by $\frac{\pi}{2}$ and π (see Theorem 3.1).

(ii) With $z \in \mathbb{C} \setminus [-e, 0]$ and $w \in \mathcal{A}$, $zwe^w = 1$, $\mathcal{A} := \{w \in \mathbb{C} \setminus \{0\} : w > -1 \text{ or } w = a + ib, b \in (-\pi, \pi) \setminus \{0\}, a > -b \cot b\}$ it holds that

$$Q_n(z) = \frac{n!}{\sqrt{2\pi n}} \frac{1}{w^n} \exp\left\{\frac{n}{w}(1 - e^{-w})\right\} (1+w)^{-1/2} \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right),$$

where the \mathcal{O} -term holds uniformly on every compact subset of $\mathbb{C} \setminus [-e, 0]$ (see Theorems 3.2 and 3.3).

In addition, we investigate the turning point $-e$, which occurs in the interval in (i) by tending $\phi \rightarrow 0$. Thereby, we get an Airy-asymptotics, which is called a strong asymptotics, as well as the above mentioned asymptotics.

In addition to this work, the position of the zeros of Q_n is investigated in [4].

2. AUXILIARY RESULTS

2.1. Technical Lemmas

LEMMA 2.1. (i) Let φ be in $(0, \frac{\pi}{2})$; then $1 < \frac{\varphi}{\sin \varphi} < \frac{\pi}{2}$.

(ii) Let φ be in $(0, \pi)$ and α in $[0, \pi - \varphi]$; then $\cos \alpha \frac{\varphi}{\sin \varphi} > -1$.

Proof. Both (i) and (ii) can be easily checked. ■

LEMMA 2.2. Let a, b be in \mathbb{R} , $a < b$, $f: [a, b] \rightarrow \mathbb{R}$, $f \in C^2[a, b]$ with $f(a) \geq f(b)$ and $f''(x) < 0$ for all x in (a, b) ; then for all x in (a, b) $f(x)$ is greater than $f(b)$.

Proof. It is $f''(x) < 0$ for all x in (a, b) ; i.e. f is strictly concave on $[a, b]$. If we assume that there is an x_0 in (a, b) with $f(x_0) \leq f(b)$, there is also a λ in $(0, 1)$ satisfying:

$$f(b) \geq f(x_0) = f(\lambda a + (1 - \lambda)b) > \lambda f(a) + (1 - \lambda)f(b) \geq f(b).$$

This is a contradiction and thus the lemma is proved. ■

2.2. The Solution of the Saddle Point Equation

To apply the saddle point method to the integral (1.2) it is required (see [7, p. 127]) that the derivative of $p(t) = \ln t$ has a simple zero on the curve γ_0 . Because this is not possible, the function $Q_n(z) = P_n(nz)$ is introduced. By (1.3) this leads to

$$p(t) = \ln t - z(e^t - 1) \quad \text{and} \quad p'(t) = \frac{1}{t} - ze^t; \quad (2.1)$$

the logarithm will be defined below. To determine t with $p'(t) = 0$, which is equivalent to solving the equation $zte^t = 1$ with $z \in \mathbb{C} \setminus \{0\}$, we refer to [5], where a similar problem (namely to solve $z(t-1)e^t = 1$) is discussed in detail. The obtained solutions and characteristics can be transferred easily and so the following results hold:

LEMMA 2.3. Define

$$\mathcal{A} := \{w \in \mathbb{C} \setminus \{0\} : w > -1 \text{ or } w = a + ib, b \in (-\pi, \pi) \setminus \{0\}, a > -b \cot b\},$$

$$\Gamma_+ := \partial \mathcal{A} \cap \{w \in \mathbb{C} : \Im(w) > 0\},$$

$$\Gamma_- := \partial \mathcal{A} \cap \{w \in \mathbb{C} : \Im(w) < 0\},$$

$$\Gamma := \partial \mathcal{A} \setminus \{0\} = \Gamma_+ \cup \Gamma_- \cup \{-1\} \quad \text{and} \quad \Psi: \overline{\mathcal{A}} \setminus \{0\} \rightarrow \mathbb{C},$$

$$\Psi(w) := \frac{1}{w} e^{-w}, \quad \text{then:}$$

(i) The equation $x = \Psi(w)$, $x \in (-e, 0)$, has exactly two solutions $w = a \pm ib$, $w \in \mathcal{A}$, which satisfy

(a) $a > -1$, $b \in (0, \pi)$, $a = -b \cot b$, $a + ib \in \Gamma_+$, $a - ib \in \Gamma_-$,

(b) $x = h(b) := -\frac{\sin b}{b} e^{b \cot b}$,

(c) $\lim_{b \rightarrow 0} h(b) = -e$, $\lim_{b \rightarrow \pi} h(b) = 0$, $h(b)$ is increasing strictly, $b \in (0, \pi)$,

(d) with $x \in (-e, 0)$ $a = a(x)$ is increasing strictly, $a((-e, 0)) = (-1, \infty)$, and $b = b(x)$ is increasing strictly, $b((-e, 0)) = (0, \pi)$,

(e) $w \in \Gamma_+$, $w = a + ib$, or $w \in \Gamma_-$, $w = a - ib$, satisfies $|w|^2 = \frac{b^2}{\sin^2 b}$.

(ii) Ψ maps \mathcal{A} conformally onto $\mathbb{C} \setminus [-e, 0]$.

(iii) Ψ maps both Γ_+ and Γ_- one-one onto $(-e, 0)$.

Proof. See [5, pp. 346–350]. ■

Due to these results, an inverse function $\Phi: \mathbb{C} \setminus \{-e, 0\} \rightarrow \mathcal{A} \cup \Gamma_+$ of Ψ can be defined (see Fig. 1):

$$\Phi(z) := \begin{cases} w, w \in \mathcal{A}, z w e^w = 1, & z \in \mathbb{C} \setminus [-e, 0] \\ w, w \in \Gamma_+, z w e^w = 1, & z \in (-e, 0). \end{cases} \quad (2.2)$$

The function Φ is analytic on $\mathbb{C} \setminus [-e, 0]$ and maps $\mathbb{C} \setminus [-e, 0]$ conformally onto \mathcal{A} . Especially, it holds for $\Phi((-\infty, -e)) = (-1, 0)$ and $\Phi((0, \infty)) = (0, \infty)$. Further, Φ solves the saddle point equation:

LEMMA 2.4. For $z \in \mathbb{C} \setminus \{-e, 0\}$ $p'(t)$ has a simple zero at $w = \Phi(z)$.

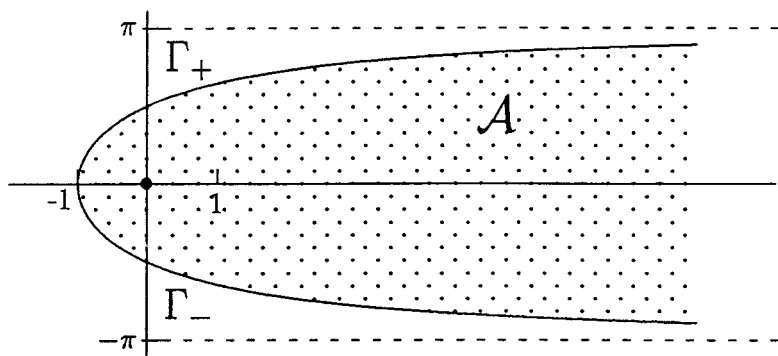


FIG. 1. The domain \mathcal{A} .

Proof. Because of (2.2), $p'(w) = 0$, so we have to show $p''(w) \neq 0$:

$$p''(w) = -\frac{1}{w^2} - ze^w = -\frac{1}{w} \left(\frac{1}{w} + 1 \right) \neq 0, \quad \text{because } -1 \notin \mathcal{A} \cup \Gamma_+. \quad \blacksquare$$

For $z \in (-e, 0)$ we introduce a parametrization (compare Lemma 2.3(i)(a), (b))

$$z = x(\phi) := -\frac{\sin(\phi)}{\phi} e^{\phi \cot \phi}, \quad \phi \in (0, \pi), \quad (2.3)$$

so that w is given as:

$$w = w(\phi) := -\phi \cot \phi + i\phi = -\frac{\phi}{\sin \phi} e^{-i\phi} = \frac{\phi}{\sin \phi} e^{i(\pi - \phi)}. \quad (2.4)$$

The next problem is to determine a curve γ_0 having w as an interior point and satisfying the condition (see [7, p.127]) that the real part of $p(t) - p(w)$ is positive for all $t \in \gamma_0 \setminus \{w\}$. We will prove that γ_0 may be chosen as a circle for z in the cut plane $\mathbb{C} \setminus [-e, 0]$ and as a semi-circle for $z \in (-e, 0)$.

3. PLANCHEREL-ROTACH ASYMPTOTICS

3.1. The Oscillating Asymptotics

For $z \in (-e, 0)$, from (1.3) we get the representation

$$\begin{aligned} Q_n(z) &= \frac{n!}{2\pi i} \int_{\gamma_0} e^{-n(\ln t - z(e^t - 1))} \frac{dt}{t} \\ &= \frac{n!}{\pi} \Im \left\{ \int_{\gamma_0^+} e^{-n(\ln t - z(e^t - 1))} \frac{dt}{t} \right\}, \end{aligned} \quad (3.1)$$

with γ_0^+ the upper half of the circle with radius $|w|$ and $\ln t = \ln |t| + i\text{ph}(t)$, $\text{ph}(t) \in [0, \pi]$. If we want to apply the saddle point method to (3.1), we have to verify the real part condition, i.e., $\Re\{p(t) - p(w)\}$ is greater than 0 for all $t \in \gamma_0^+$. By using (2.3) and (2.4) with $r(\phi) = \frac{\phi}{\sin \phi}$, $t = t(\psi) := r(\phi) e^{i\psi}$, $\psi \in [0, \pi]$, $w(\phi) = t(\pi - \phi)$, and $R(\psi) := \Re\{p(t(\psi)) - p(t(\pi - \phi))\}$ we will prove that $R(\psi)$ is greater than 0 for all $\psi \in [0, \pi] \setminus \{\pi - \phi\}$.

(a) Computation of $R(\psi)$ and $R'(\psi) = \frac{dR(\psi)}{d\psi}$:

$$\begin{aligned} R(\psi) &= \Re\{p(t(\psi)) - p(t(\pi - \phi))\} \\ &= \Re\{\ln(r(\phi) e^{i\psi}) - x(\phi) e^{r(\phi) e^{i\psi}} - \ln(r(\phi) e^{i(\pi - \phi)}) + x(\phi) e^{r(\phi) e^{i(\pi - \phi)}}\} \\ &= -x(\phi) e^{r(\phi) \cos \psi} \cos(r(\phi) \sin \psi) - \frac{1}{r(\phi)} \cos(\phi). \end{aligned} \quad (3.2)$$

That gives:

$$\begin{aligned} R'(\psi) &= -x(\phi) e^{r(\phi) \cos \psi} r(\phi) (-\sin \psi) \cos(r(\phi) \sin \psi) \\ &\quad - x(\phi) e^{r(\phi) \cos \psi} (-\sin(r(\phi) \sin \psi)) r(\phi) \cos \psi \\ &= x(\phi) e^{r(\phi) \cos \psi} r(\phi) \sin(\psi + r(\phi) \sin \psi). \end{aligned} \quad (3.3)$$

With $f(\psi) := \psi + r(\phi) \sin \psi$ and $g(\psi) := x(\phi) e^{r(\phi) \cos \psi} r(\phi)$ we have:

$$\begin{aligned} R'(\psi) &= g(\psi) \sin(f(\psi)), \\ g(\psi) &< 0 \quad \text{for all } \psi \in [0, \pi] \quad \text{and} \\ R'(\psi) &= 0 \quad \text{if and only if } f(\psi) = k\pi, \quad k \in \mathbb{N}_0. \end{aligned} \quad (3.4)$$

(b) Proof that $R(\psi)$ is greater than 0 for all ψ in $[0, \pi - \phi)$: Because $f(0) = 0$, $f(\pi - \phi) = \pi$, and $f'(\psi) = 1 + \frac{\phi}{\sin \phi} \cos \psi > 0$ for $\psi \in (0, \pi - \phi)$ (cf. Lemma 2.1(ii)), by using (3.4) it follows that $R'(\psi) < 0$ for $\psi \in (0, \pi - \phi)$. Since $R(\pi - \phi) = 0$, the allegation is proved.

(c) Proof that $R(\psi)$ is greater than 0 for all ψ in $(\pi - \phi, \pi]$:

Case 1. $\phi \leq \frac{\pi}{2}$. It is sufficient to show that $f(\psi) \in (\pi, 2\pi)$, for $\psi \in (\pi - \phi, \pi)$, then from (3.4) it follows that $R'(\psi) > 0$ for $\psi \in (\pi - \phi, \pi)$. First, by Lemma 2.1(i) it holds that $f(\psi) = \psi + r(\phi) \sin \psi < \pi + \frac{\pi}{2} < 2\pi$. On the other hand it holds that $f(\pi - \phi) = f(\pi) = \pi$, $f'(\psi) = 1 + \cos \psi \frac{\phi}{\sin \phi}$ and $f''(\psi) = -\sin \psi \frac{\phi}{\sin \phi} < 0$ for $\psi \in (\pi - \phi, \pi)$. And thus Lemma 2.2 gives $f(\psi) > \pi$.

Case 2. $\phi > \frac{\pi}{2}$. Because $R(\pi) = x(\phi) e^{r(\phi)(-1)} \cos(0) - \frac{1}{r(\phi)} \cos(\phi) > 0$ and $R(\pi - \phi) = 0$, it is sufficient to show that

$$R(\psi_0) > 0, \quad \text{for all } \psi_0 \in \mathcal{N}, \quad \mathcal{N} := \{\psi \in (\pi - \phi, \pi) : R'(\psi) = 0\}.$$

Since $f(\psi_0) > 0$ and by (3.4), it follows that for $\psi_0 \in \mathcal{N}$ $f(\psi_0) \in \{k\pi : k \in \mathbb{N}\}$ if and only if there is a $k_0 \in \mathbb{N}$ with $\psi_0 + r(\phi) \sin \psi_0 = k_0\pi$, i.e. $r(\phi) \sin \psi_0 = k_0\pi - \psi_0$.

$\alpha.$ $\psi_0 \in (\pi - \phi, \phi)$, by (3.2) it follows that

$$\begin{aligned} R(\psi_0) &= -x(\phi) e^{r(\phi) \cos \psi_0} \cos(r(\phi) \sin \psi_0) + x(\phi) e^{r(\phi) \cos(\pi - \phi)} \cos \phi \\ &= x(\phi) e^{r(\phi) \cos(\pi - \phi)} \cos \phi \\ &\quad - x(\phi) e^{r(\phi) \cos \psi_0} \begin{cases} \cos \psi_0, & k_0 = 2m, m \in \mathbb{N} \\ -\cos \psi_0, & k_0 = 2m - 1, m \in \mathbb{N} \end{cases} \\ &= x(\phi) \left(e^{r(\phi) \cos(\pi - \phi)} \cos \phi - e^{r(\phi) \cos \psi_0} \begin{Bmatrix} \cos \psi_0 \\ -\cos \psi_0 \end{Bmatrix} \right) > 0, \end{aligned}$$

because $\cos(\pi - \phi) = -\cos \phi > |\cos \psi_0| \geq 0$, $\psi_0 \in (\pi - \phi, \phi)$, $\phi > \frac{\pi}{2}$.

$\beta.$ $\psi_0 \in [\phi, \pi)$. This case does not exist, because $\mathcal{N} \cap [\phi, \pi)$ is empty, which is proved as follows:

First, it holds that $f(\phi) = \phi + r(\phi) \sin \phi = 2\phi > \pi$, $f(\pi) = \pi$ and $f''(\psi) = -r(\phi) \sin \psi < 0$ for $\psi \in (\phi, \pi)$. Then Lemma 2.2 with $a = \phi$ and $b = \pi$ gives:

$$f(\psi) > f(\pi) = \pi, \quad \text{for all } \psi \in [\phi, \pi).$$

Second, for $\psi \in [\phi, \pi)$ it follows that

$$f(\psi) = \psi + \frac{\sin \psi}{\sin \phi} \phi < \pi + 1\pi = 2\pi.$$

That means that $f(\psi) \in (\pi, 2\pi)$ for $\psi \in [\phi, \pi)$ and it follows by (3.4) that $R'(\psi) \neq 0$. ■

Thus the real part condition is accomplished and the saddle point method may be applied.

THEOREM 3.1. *Let $x(\phi)$ be defined by (2.3). Then for $\phi \in (0, \pi)$, as $n \rightarrow \infty$,*

$$Q_n(x(\phi)) = k_n(\phi) \left(\sin \left(n \left(\pi - \phi + \frac{\sin^2 \phi}{\phi} \right) + \eta(\phi) \right) + \mathcal{O} \left(\frac{1}{n} \right) \right),$$

with $\arccos: [-1, 1] \rightarrow [0, \pi]$, $\eta: (0, \pi) \rightarrow (\frac{\pi}{2}, \pi)$, $k_n(\phi): (0, \pi) \rightarrow (0, \infty)$ and

$$\begin{aligned} k_n(\phi) &:= \frac{n!}{\sqrt{\pi n}} \sqrt{2} e^{-n(\ln(\phi/\sin \phi) + x(\phi) + ((\sin \phi \cos \phi)/\phi))} \\ &\quad \times \left(\left(\frac{\phi}{\sin \phi} - \cos \phi \right)^2 + \sin^2 \phi \right)^{-1/4} \\ \eta(\phi) &:= \frac{\pi}{2} + \frac{1}{2} \arccos \left(\frac{1 - \phi \cot(\phi)}{((1 - \phi \cot(\phi))^2 + \phi^2)^{1/2}} \right). \end{aligned}$$

Proof. According to [7, p. 127] with (2.4), (3.1), and Lemma 2.4 we obtain

$$\begin{aligned} Q_n(x(\phi)) &= \frac{n!}{\pi} \Im \left\{ \int_{\gamma_0^\pm} e^{-n(\ln t - x(\phi)(e^t - 1))} \frac{dt}{t} \right\} \\ &= \frac{n!}{\pi} \Im \left\{ 2e^{-np(w(\phi))} \frac{1}{\sqrt{n}} \left(\Gamma\left(\frac{1}{2}\right) a_0 + \mathcal{O}\left(\frac{1}{n}\right) \right) \right\}, \end{aligned}$$

with $a_0 = (w(\phi) \sqrt{2p''(w(\phi))})^{-1}$ and $\omega_0 := \text{ph}(p''(w(\phi)))$ satisfying $|\omega_0 + 2\omega| \leq \frac{\pi}{2}$, where ω is the limiting value of $\text{ph}(t - w(\phi))$ as $t \rightarrow w(\phi)$ along $(w(\phi), -|w(\phi)|)$, which means the part of γ_0^+ between $-|w(\phi)|$ and $w(\phi)$.

(a) Computation of $p(w(\phi))$ gives:

$$\begin{aligned} p(w(\phi)) &= p(t(\pi - \phi)) = \ln t(\pi - \phi) - x(\phi)(e^{t(\pi - \phi)} - 1) \\ &= \ln r(\phi) + i(\pi - \phi) + x(\phi) \\ &\quad + \frac{\sin \phi}{\phi} e^{\phi \cot \phi} e^{(\phi/\sin \phi)(\cos(\pi - \phi) + i \sin(\pi - \phi))} \\ &= \ln \frac{\phi}{\sin \phi} + x(\phi) + \frac{\sin \phi \cos \phi}{\phi} + i \left(\pi - \phi + \frac{\sin^2 \phi}{\phi} \right). \end{aligned} \quad (3.5)$$

(b) With $t \rightarrow w(\phi)$ along $(w(\phi), -|w(\phi)|)$, ω is given by:

$$\begin{aligned} \omega &= \lim_{t \rightarrow w(\phi)} \text{ph}(t - w(\phi)) = \lim_{\psi \rightarrow (\pi - \phi), \psi > (\pi - \phi)} \text{ph}(r(\phi)(e^{i\psi} - e^{i(\pi - \phi)})) \\ &= \pi - \phi + \lim_{\varepsilon \rightarrow 0, \varepsilon > 0} \text{ph}(e^{i\varepsilon} - 1) = \frac{3}{2} \pi - \phi. \end{aligned} \quad (3.6)$$

(c) Since $x(\phi) w(\phi) e^{w(\phi)} = 1$ and due to Lemma 2.4, ω_0 is computed as follows,

$$\begin{aligned} \omega_0 &= \text{ph}(p''(w(\phi))) = \text{ph} \left(-\frac{1}{w(\phi)^2} (1 + w(\phi)) \right) \\ &= \text{ph} \left(\frac{e^{-i\pi} e^{-2i(\pi - \phi)}}{(r(\phi))^2} \right) + \text{ph}(1 + w(\phi)) = -3\pi + 2\phi + \text{ph}(1 + w(\phi)), \end{aligned}$$

and since $\Re(w(\phi)) > -1$, the condition $|\omega_0 + 2\omega| \leq \frac{\pi}{2}$ is satisfied. With $w(\phi) = r(\phi) e^{i(\pi - \phi)}$ and $\arccos : [-1, 1] \rightarrow [0, \pi]$, ω_0 holds further:

$$\omega_0 = -3\pi + 2\phi + \arccos \left(\frac{1 - \phi \cot(\phi)}{((1 - \phi \cot(\phi))^2 + \phi^2)^{1/2}} \right). \quad (3.7)$$

(d) Computation of a_0 gives:

$$\begin{aligned}
 a_0 &= (w(\phi) \sqrt{2p''(w(\phi))})^{-1} \\
 &= \frac{1}{\sqrt{2}} \frac{e^{-i(\pi-\phi)}}{r(\phi)} e^{-i(\omega_0/2)} \left| \frac{1}{r(\phi) e^{i(\pi-\phi)}} \left(\frac{1}{r(\phi) e^{i(\pi-\phi)}} + 1 \right) \right|^{-1/2} \\
 &= \frac{1}{\sqrt{2}} e^{-i(\pi-\phi+(\omega_0/2))} |e^{-i(\pi-\phi)} + r(\phi)|^{-1/2} \\
 &= \frac{1}{\sqrt{2}} e^{-i(\pi-\phi+(\omega_0/2))} \left[\left(\frac{\phi}{\sin \phi} - \cos \phi \right)^2 + \sin^2 \phi \right]^{-1/4}. \tag{3.8}
 \end{aligned}$$

Altogether, from (3.5), (3.7), and (3.8) we obtain:

$$\begin{aligned}
 Q_n(x(\phi)) &= \frac{n!}{\sqrt{\pi n}} \mathfrak{I} \left\{ 2e^{-n(\ln(\phi/\sin \phi) + x(\phi) + ((\sin \phi \cos \phi)/\phi) + i(\pi - \phi + (\sin^2 \phi/\phi)))} \right. \\
 &\quad \times \left(\frac{1}{\sqrt{2}} e^{-i(\pi-\phi+(\omega_0/2))} \left[\left(\frac{\phi}{\sin \phi} - \cos \phi \right)^2 + \sin^2 \phi \right]^{-1/4} \right. \\
 &\quad \left. \left. + \mathcal{O}(n^{-1}) \right) \right\} \\
 &= \frac{n!}{\sqrt{\pi n}} \sqrt{2} e^{-n(\ln(\phi/\sin \phi) + x(\phi) + ((\sin \phi \cos \phi)/\phi))} \\
 &\quad \times \left[\left(\frac{\phi}{\sin \phi} - \cos \phi \right)^2 + \sin^2 \phi \right]^{-1/4} \\
 &\quad \times \left(\sin \left(\phi - \pi - \frac{\omega_0}{2} - n \left(\pi - \phi + \frac{\sin^2 \phi}{\phi} \right) \right) + \mathcal{O}(n^{-1}) \right) \\
 &= \frac{n!}{\sqrt{\pi n}} \sqrt{2} e^{-n(\ln(\phi/\sin \phi) + x(\phi) + ((\sin \phi \cos \phi)/\phi))} \\
 &\quad \times \left[\left(\frac{\phi}{\sin \phi} - \cos \phi \right)^2 + \sin^2 \phi \right]^{-1/4} \left(\sin \left(n \left(\pi - \phi + \frac{\sin^2 \phi}{\phi} \right) \right) \right. \\
 &\quad \left. + \frac{\pi}{2} + \frac{1}{2} \arccos \left(\frac{1 - \phi \cot(\phi)}{((1 - \phi \cot(\phi))^2 + \phi^2)^{1/2}} \right) + \mathcal{O}(n^{-1}) \right). \blacksquare
 \end{aligned}$$

3.2. The Asymptotics on the Cut Plane

For $z \in \mathbb{C} \setminus [-e, 0]$ we will apply the saddle point method to (1.3). Therefore we will choose γ_0 as a circle with radius $|w|$. The logarithm is

defined by $\ln t = \ln |t| + i \operatorname{ph}(t)$ with $\operatorname{ph}(t) \in [0, 2\pi]$ for $z \in \mathbb{C} \setminus [-e, \infty)$ and $\operatorname{ph}(t) \in [-\pi, \pi]$ for $z \in (0, \infty)$.

First, we have to prove the real part condition again; i.e., $\Re\{p(t) - p(w)\}$ is greater than 0 for all $t \in \gamma_0 \setminus \{w\}$. With $w = re^{i\alpha}$, $r > 0$, $\alpha \in [0, 2\pi)$, $w \in \mathcal{A}$, $r < r_{\max} := \frac{\pi - \alpha}{\sin \alpha}$ for $\alpha \in (0, \pi) \cup (\pi, 2\pi)$ ($r < r_{\max} := 1$ for $\alpha = \pi$) and $t = t(\psi) = re^{i\psi}$, $\psi \in [0, 2\pi]$, for $\alpha \in (0, 2\pi)$ ($[-\pi, \pi]$ for $\alpha = 0$), we have to show that $R(\psi) := \Re\{p(t(\psi)) - p(w)\}$ is greater than 0 for all $\psi \neq \alpha$.

By (2.1) and $zwe^w = 1$ computation of $R(\psi)$ gives:

$$\begin{aligned} R(\psi) &= \Re\{-ze^{t(\psi)} + ze^w\} = \Re\left\{\frac{1}{r}e^{-i\alpha}(1 - e^{r(e^{i\psi} - e^{i\alpha})})\right\} \\ &= \frac{1}{r} \cos \alpha - \frac{1}{r} e^{r(\cos \psi - \cos \alpha)} \cos(\alpha - r(\sin \psi - \sin \alpha)). \end{aligned} \quad (3.9)$$

If $\alpha = 0$ the allegation follows directly from (3.9). For $\alpha \neq 0$ we define

$$\begin{aligned} f(r, \psi) &:= e^{r(\cos \psi - \cos \alpha)} \cos(\alpha - r(\sin \psi - \sin \alpha)), \\ \mathcal{G} &:= \{(r, \psi) \in \mathbb{R}^2 : 0 < r < r_{\max}, \psi \in (0, 2\pi + \varepsilon)\} \\ &\quad \text{with } 0 < \varepsilon < \min\{\alpha, 2\pi - \alpha\}, \\ \mathcal{Q} &:= \{(r_0, \psi_0) \in \overline{\mathcal{G}} : f(r_0, \psi_0) \geq f(r, \psi) \text{ for all } (r, \psi) \in \overline{\mathcal{G}}\}, \quad \text{and} \\ \mathcal{M} &:= \{(r, \psi) \in \overline{\mathcal{G}} : r = 0 \text{ or } r = r_{\max}, \psi \in \{\alpha, 2\pi - \alpha\} \text{ or } \psi = \alpha\}. \end{aligned}$$

Hence, it is sufficient to show that $f(r, \psi) = \cos \alpha$ for all $(r, \psi) \in \mathcal{M}$ and $\mathcal{Q} \subset \mathcal{M}$. The first condition can be easily checked, and thus we only have to prove $\mathcal{Q} \subset \mathcal{M}$. For $(r_*, \psi_*) \in \mathcal{Q}$ it follows directly that $f(r_*, \psi_*)$ is not less than $\cos \alpha$. To show that $(r_*, \psi_*) \in \mathcal{M}$, we investigate in (a), (b), (c), and (d) possible maximums on the edge of \mathcal{G} and in (e) in the interior of \mathcal{G} .

(a) $\mathbf{r}_* = \mathbf{0}$. $(r_*, \psi_*) \in \mathcal{M}$ follows immediately.

(b) $\mathbf{r}_* = \mathbf{r}_{\max}$. To show that $\psi_* \in \{\alpha, 2\pi - \alpha\}$ we define $g(\psi) := f(r_{\max}, \psi)$, then

$$\begin{aligned} g(\psi) &= \begin{cases} e^{((\pi - \alpha)/\sin \alpha)(\cos \psi - \cos \alpha)} \cos\left(\alpha - \frac{\pi - \alpha}{\sin \alpha}(\sin \psi - \sin \alpha)\right), & \alpha \in (0, 2\pi) \setminus \{\pi\} \\ e^{\cos \psi - \cos \pi} \cos(\pi - (\sin \psi - \sin \pi)), & \alpha = \pi \end{cases} \\ &= \begin{cases} -e^{(\alpha - \pi) \cot \alpha} e^{(\pi - \alpha/\sin \alpha) \cos \psi} \cos\left(\frac{\pi - \alpha}{\sin \alpha} \sin \psi\right), & \alpha \in (0, 2\pi) \setminus \{\pi\} \\ -e^{1 + \cos \psi} \cos(\sin \psi), & \alpha = \pi, \end{cases} \end{aligned} \quad (3.10)$$

so we have to prove $g(\psi) < \cos \alpha$ for all $\psi \in [0, 2\pi + \varepsilon] \setminus \{\alpha, 2\pi - \alpha\}$.

(b1) $\alpha \in (0, \pi)$, $\psi \in [0, \pi] \setminus \{\alpha\}$. With $\alpha =: \pi - \phi$, $\phi \in (0, \pi)$, it follows from Section 3.1, especially (3.2), that $g(\psi) < \cos \alpha$.

(b2) $\alpha \in (0, \pi)$, $\psi \in (\pi, 2\pi] \setminus \{2\pi - \alpha\}$. By using $\varphi \in [0, \pi] \setminus \{\alpha\}$ with $\psi = 2\pi - \varphi$, it follows from (3.10) that $g(\psi) = g(\varphi)$ and hence by (b1), $g(\psi) < \cos \alpha$ for all $\psi \in (\pi, 2\pi] \setminus \{2\pi - \alpha\}$.

(b3) $\alpha \in (0, \pi)$, $\psi \in (2\pi, 2\pi + \varepsilon]$. By (b1) and the 2π -periodicity of g it holds that $g(\psi) < \cos \alpha$.

(b4) $\alpha = \pi$. By (3.10) we have to show that $h(\psi) := e^{\cos \psi} \cos(\sin \psi) > \frac{1}{e}$ for all $\psi \in [0, 2\pi + \varepsilon] \setminus \{\pi\}$. Since $h'(\psi) = -e^{\cos \psi} \sin(\psi + \sin \psi)$, the function h has an absolute minimum in $[0, 2\pi + \varepsilon]$ at $\psi = \pi$ and by $h(\pi) = e^{-1}$ the prove is completed.

(b5) $\alpha \in (\pi, 2\pi)$. With $\alpha = 2\pi - \beta$, $\beta \in (0, \pi)$, it follows from (3.10) that

$$g(\psi) = -e^{(\beta - \pi) \cot \beta} e^{((\pi - \beta)/\sin \beta) \cos \psi} \cos \left(\frac{\pi - \beta}{\sin \beta} \sin \psi \right).$$

Hence, we can deduce immediately from (b1)–(b3) that $g(\psi) < \cos \alpha$ for all $\psi \in [0, 2\pi + \varepsilon] \setminus \{\alpha, 2\pi - \alpha\}$.

(c) $\mathbf{r}_* \in (0, \mathbf{r}_{\max})$, $\psi_* = 0$. By the 2π -periodicity of f with reference to ψ , it follows that $(r_*, 2\pi) \in \mathcal{Q}$ also. This is investigated in (e).

(d) $\mathbf{r}_* \in (0, \mathbf{r}_{\max})$, $\psi_* = 2\pi + \varepsilon$. By the 2π -periodicity of f with reference to ψ , it follows that also $(r_*, \varepsilon) \in \mathcal{Q}$. This is investigated in (e).

(e) $\mathbf{r}_* \in (0, \mathbf{r}_{\max})$, $\psi_* \in (0, 2\pi + \varepsilon)$. We assume: $\psi_* \neq \alpha$. Because $(r_*, \psi_*) \in \mathcal{Q}$, $(r_*, \psi_*) \in \mathcal{G}$, and \mathcal{G} is open, (r_*, ψ_*) must comply with:

$$\frac{\partial f(r, \psi)}{\partial r} = \frac{\partial f(r, \psi)}{\partial \psi} = 0, \quad (r, \psi) = (r_*, \psi_*). \quad (3.11)$$

Computation of $\partial f / \partial r$ gives:

$$\begin{aligned} \frac{\partial f(r, \psi)}{\partial r} &= e^{r(\cos \psi - \cos \alpha)} ((\cos \psi - \cos \alpha) \cos(\alpha - r(\sin \psi - \sin \alpha)) \\ &\quad - \sin(\alpha - r(\sin \psi - \sin \alpha))(-(\sin \psi - \sin \alpha))) \\ &= -2e^{r(\cos \psi - \cos \alpha)} \left(\sin \left(\frac{\psi - \alpha + 2r(\sin \psi - \sin \alpha)}{2} \right) \sin \left(\frac{\psi - \alpha}{2} \right) \right). \end{aligned}$$

That means $f_r(r_*, \psi_*) = 0$ if and only if $\psi_* - \alpha = 2k\pi$, $k \in \mathbb{Z}$ or $\psi_* - \alpha + 2r_*(\sin \psi_* - \sin \alpha) = 2k\pi$, $k \in \mathbb{Z}$.

Case 1. $\psi_* - \alpha = 2k\pi$, $k \in \mathbb{Z}$, by $\psi_* = \alpha + 2k\pi$; it follows that

if $k = 0$: $\psi_* = \alpha$, contradiction to the assumption!

if $k < 0$: $\psi_* < 0$, contradiction to $\psi_* \in \mathcal{G}$!

if $k > 0$: $\psi_* = 2k\pi + \alpha > 2\pi + \varepsilon$, contradiction to $\psi_* \in \mathcal{G}$!

Case 2. $\psi_* - \alpha + 2r_*(\sin \psi_* - \sin \alpha) = 2k\pi$, $k \in \mathbb{Z}$; that means:

$$r_*(\sin \psi_* - \sin \alpha) = k\pi + \frac{\alpha - \psi_*}{2}. \quad (3.12)$$

By

$$\begin{aligned} \frac{\partial f(r, \psi)}{\partial \psi} &= e^{r(\cos \psi - \cos \alpha)} ((-r \sin \psi) \cos(\alpha - r(\sin \psi - \sin \alpha)) \\ &\quad - \sin(\alpha - r(\sin \psi - \sin \alpha))(-r \cos \psi)) \\ &= -re^{r(\cos \psi - \cos \alpha)} (\sin(\psi - \alpha + r(\sin \psi - \sin \alpha))), \end{aligned}$$

and (3.11), it follows that $\psi_* - \alpha + r_*(\sin \psi_* - \sin \alpha) = m\pi$, $m \in \mathbb{Z}$, and hence by (3.12), $\psi_* - \alpha + k\pi + \frac{\alpha - \psi_*}{2} = m\pi$. That means $\psi_* = \alpha + 2(m - k)\pi$ and we can deduce:

if $m = k$: $\psi_* = \alpha$, a contradiction to the assumption!

if $m < k$: $\psi_* < 0$, a contradiction to $\psi_* \in \mathcal{G}$!

if $m > k$: $\psi_* > 2\pi + \varepsilon$, a contradiction to $\psi_* \in \mathcal{G}$!

Hence, we have shown that neither Case 1 nor Case 2 can occur and our assumption must be wrong. ■

Thus, the real part condition is accomplished and the saddle point method may be applied.

THEOREM 3.2. *Let z be in $\mathbb{C} \setminus [-e, 0]$ and $\Phi(z) = w \in \mathcal{A}$, $zwe^w = 1$. Then, as $n \rightarrow \infty$:*

$$Q_n(z) = \frac{n!}{\sqrt{2\pi n}} \frac{1}{w^n} \exp \left\{ \frac{n}{w} (1 - e^{-w}) \right\} (1 + w)^{-1/2} \left(1 + \mathcal{O} \left(\frac{1}{n} \right) \right).$$

Proof. According to [7, p. 127] and (3.1) it follows that

$$\begin{aligned} Q_n(z) &= \frac{n!}{2\pi i} \int_{\gamma_0} e^{-n(\ln t - z(e^t - 1))} \frac{dt}{t} \\ &= \frac{n!}{\pi i} e^{-np(w)} \frac{1}{\sqrt{n}} \left(\Gamma \left(\frac{1}{2} \right) a_0 + \mathcal{O} \left(\frac{1}{n} \right) \right), \end{aligned}$$

with $a_0 = (w \sqrt{2p''(w)})^{-1}$ and $\omega_0 := \text{ph}(p''(w))$ satisfying $|\omega_0 + 2\omega| \leq \frac{\pi}{2}$, where ω is the limiting value of $\text{ph}(t - w)$ as $t \rightarrow w$ along the part of γ_0 between w and the endpoint of γ_0 .

(a) By $zwe^w = 1$ computation of $p(w)$ gives:

$$p(w) = \ln w - z(e^w - 1) = \ln w - \frac{1}{w} + \frac{e^{-w}}{w}. \quad (3.13)$$

(b) With $w = |w| e^{i\alpha}$ and $t = |w| e^{i\psi}$, ω is given by:

$$\begin{aligned} \omega &= \lim_{t \rightarrow w} \text{ph}(t - w) = \lim_{\psi \rightarrow \alpha, \psi > \alpha} \text{ph}(|w| e^{i\psi} - |w| e^{i\alpha}) \\ &= \lim_{\varepsilon \rightarrow 0, \varepsilon > 0} \text{ph}(|w| e^{i\alpha} (e^{i\varepsilon} - 1)) = \alpha + \frac{\pi}{2}. \end{aligned} \quad (3.14)$$

(c) By $zwe^w = 1$, $w = |w| e^{i\alpha}$, and Lemma 2.4, ω_0 is computed as follows

$$\begin{aligned} \omega_0 &= \text{ph}(p''(w)) = \text{ph}\left(-\frac{1}{w^2}(1+w)\right) \\ &= \text{ph}\left(e^{-i\pi} \frac{1}{|w|^2} e^{-2i\alpha}(1+w)\right) = -\pi - 2\alpha + \text{ph}(1+w), \end{aligned} \quad (3.15)$$

and since $\Re(w) > -1$, the condition $|\omega_0 + 2\omega| \leq \frac{\pi}{2}$ is satisfied.

(d) Computation of a_0 gives

$$\begin{aligned} a_0 &= (w \sqrt{2p''(w)})^{-1} = \frac{1}{\sqrt{2}} \frac{e^{-i\alpha}}{|w|} \left| -\frac{1}{w^2}(1+w) \right|^{-1/2} e^{-(1/2)i\omega_0} \\ &= \frac{1}{\sqrt{2}} |1+w|^{-1/2} e^{i(\pi/2)} e^{-(i/2)\text{ph}(1+w)} = \frac{i}{\sqrt{2}} (1+w)^{-1/2}, \end{aligned} \quad (3.16)$$

with $\ln(1+w) = \ln|1+w| + i\text{ph}(1+w)$, $|\text{ph}(1+w)| \leq \frac{\pi}{2}$.

From (3.13), (3.15), and (3.16) we now obtain:

$$\begin{aligned} Q_n(z) &= \frac{n!}{\pi i} \exp \left\{ -n \left(\ln w - \frac{1}{w} + \frac{e^{-w}}{w} \right) \right\} \\ &\quad \times \left(\sqrt{\frac{\pi}{n}} \frac{i}{\sqrt{2}} (1+w)^{-1/2} + \mathcal{O}(n^{-3/2}) \right) \\ &= \frac{n!}{\sqrt{2\pi n}} \frac{1}{w^n} \exp \left\{ \frac{n}{w} (1 - e^{-w}) \right\} ((1+w)^{-1/2} + \mathcal{O}(n^{-1})). \quad \blacksquare \end{aligned}$$

The result of Theorem 3.2 can be strengthened: the asymptotics is valid uniformly in every compact set K lying in $\mathbb{C} \setminus [-e, 0]$.

THEOREM 3.3. *Let K be a compact subset of $\mathbb{C} \setminus [-e, 0]$ and*

$$G_n(z) := \frac{n!}{\sqrt{2\pi n}} \frac{1}{w^n} \exp \left\{ \frac{n}{w} (1 - e^{-w}) \right\} (1+w)^{-1/2},$$

$$z \in \mathbb{C} \setminus [-e, 0], \quad w = \Phi(z).$$

Then $Q_n(z)/G_n(z)$ converges uniformly to 1 on K .

Proof. Since the saddle point method is a special application of Laplace's method, the following proof is based on the proof of this method (see [7, pp. 121–125]) with the difference that the function $p(t)$ depends on a parameter z besides:

$$p(z, t) := \ln t - z(e^t - 1), \quad t \in \mathbb{C} \setminus \{0\}, \quad z \in K.$$

The branch of the logarithm must be chosen in a suitable way, so that p is analytic in the sets appearing below (therefore K has to be split into two compact sets, if necessary). Further, we must mention that if s is odd, the coefficients used below a_s disappear when the Laplace method is replaced by the saddle point method. This happens because a different ω_0 (see (3.15)) has to be chosen (compare with the variable v introduced below with $\text{ph}(v) = \omega_0$).

(a) Because $z \in K$, K compact, there is an $M_1 > 0$ with $|w|^{-1} = |\Phi(z)|^{-1} \leq M_1$ for all $z \in K$. Further, the functions p and $q(t) = t^{-1}$ have the following power series representations in a neighborhood of w :

$$p(z, t) = p(z, w) + \sum_{s=0}^{\infty} p_s(z)(t-w)^{s+\mu}, \quad \mu = 2,$$

$$p_s(z) = \frac{1}{(s+2)!} \frac{\partial^{(s+2)} p}{\partial t^{(s+2)}}(z, w), \quad \frac{\partial^s p}{\partial t^s}(z, w) = \frac{(-1)^{s-1} (s-1)!}{w^s} - \frac{1}{w},$$

$$q(t) = \sum_{s=0}^{\infty} q_s(t-w)^{s+\lambda-1}, \quad q_s = \frac{(-1)^s}{w^{s+1}}, \quad \lambda = 1. \quad (3.17)$$

Hence, we obtain the following estimations for $|p_s(z)|$ and $|p_0(z)|$, which hold for all $z \in K$:

$$|p_s(z)| \leq \frac{1}{(s+2)!} ((s+1)! M_1^{s+2} + M_1) \leq (M_1 + 1)^{s+2} =: M_2^{s+2}, \quad (3.18)$$

$$|p_0(z)| = \left| \frac{1}{2} \left(-\frac{1}{w^2} - \frac{1}{w} \right) \right| \geq M_3 > 0, \quad (3.19)$$

$$\frac{1}{|p_0(z)|} \leq M_4, \quad M_4 := \max \left\{ 1, \frac{1}{M_3} \right\}. \quad (3.20)$$

(b) We set $u = g(z, t) := \sqrt{p(z, t) - p(z, w)}$ with $\ln u = \ln |u| + i \operatorname{ph}(u)$, $\operatorname{ph}(u) \in [-\pi, \pi]$; then (3.17) gives:

$$\begin{aligned} u &= \left(\sum_{s=0}^{\infty} p_s(z)(t-w)^{s+2} \right)^{1/2} \\ &= \left(p_0(z)(t-w)^2 \left(1 + \sum_{s=1}^{\infty} \frac{p_s(z)}{p_0(z)} (t-w)^s \right) \right)^{1/2} \\ &= \sqrt{p_0(z)} (t-w) \left(1 + \sum_{k=1}^{\infty} \binom{1/2}{k} \left(\sum_{s=1}^{\infty} \frac{p_s(z)}{p_0(z)} (t-w)^s \right)^k \right) \\ &= \sqrt{p_0(z)} (t-w) \\ &\quad \times \left(1 + \sum_{m=1}^{\infty} (t-w)^m \left(\sum_{k=1}^m \binom{1/2}{k} \sum_{\substack{v_1 + \dots + v_k = m \\ v_i \geq 1}} \prod_{i=1}^k \frac{p_{v_i}(z)}{p_0(z)} \right) \right). \end{aligned}$$

(c) Let $g(z, t) = \sum_{m=0}^{\infty} g_m(z)(t-w)^{m+1}$; then there is an M_6 greater than 0 with $|g_m(z)| \leq M_2 M_6^m$ for all $z \in K$.

Proof. The case $m=0$ follows immediately from $g_0(z) = \sqrt{p_0(z)}$ and (3.18), so we consider $m \in \mathbb{N}$, $1 \leq k \leq m$. By (3.18) and (3.20) it follows that

$$\left| \prod_{i=1}^k \frac{p_{v_i}(z)}{p_0(z)} \right| \leq \prod_{i=1}^k M_2^{v_i+2} M_4 \leq M_4^k M_2^{m+2k} \leq (M_4 M_2^3)^m =: M_5^m.$$

Since $|\binom{1/2}{k}| \leq 1$ and $\sum_{v_1 + \dots + v_k = m, v_i \geq 1} 1 = \binom{m-1}{k-1}$, we obtain:

$$\begin{aligned} |g_m(z)| &= \left| \sqrt{p_0(z)} \sum_{k=1}^m \binom{1/2}{k} \sum_{\substack{v_1 + \dots + v_k = m \\ v_i \geq 1}} \prod_{i=1}^k \frac{p_{v_i}(z)}{p_0(z)} \right| \\ &\leq M_2 \sum_{k=1}^m \binom{m-1}{k-1} M_5^m \\ &= M_2 M_5^m \sum_{k=0}^{m-1} \binom{m-1}{k} \leq M_2 (2M_5)^m =: M_2 M_6^m. \quad (3.21) \end{aligned}$$

Especially, we can conclude that $g(z, t)$ converges in the circle $\{t: |t - w| < M_6^{-1}\}$ for all $z \in K$.

(d) We define $n_0 := \max\{3, 1 + 3M_2(M_3)^{-1/2}\}$; then for all z in K $g(z, t)$ maps $\{t: |t - w| < (n_0 M_6)^{-1}\}$ conformally onto a domain U with $0 \in U$.

Proof. Let z be in K , $|t_i - w| < (n_0 M_6)^{-1}$, $i = 1, 2$, $t_1 \neq t_2$; then by (3.19) and (c) it follows that $|g(z, t_1) - g(z, t_2)|$

$$\begin{aligned} &= \left| g_0(z)(t_1 - t_2) + (t_1 - t_2) \sum_{m=2}^{\infty} g_{m-1}(z) \frac{(t_1 - w)^m - (t_2 - w)^m}{(t_1 - w) - (t_2 - w)} \right| \\ &= \left| g_0(z)(t_1 - t_2) + (t_1 - t_2) \sum_{m=2}^{\infty} g_{m-1}(z) \sum_{v=0}^{m-1} (t_1 - w)^{m-1-v} (t_2 - w)^v \right| \\ &\geq |t_1 - t_2| \left(\sqrt{M_3} - \sum_{m=2}^{\infty} M_2 M_6^{m-1} m \left(\frac{1}{n_0 M_6} \right)^{m-1} \right) \\ &= |t_1 - t_2| \left(\sqrt{M_3} - M_2 \frac{2n_0 - 1}{(n_0 - 1)^2} \right) \\ &> |t_1 - t_2| \left(\sqrt{M_3} - 3M_2 \frac{1}{n_0 - 1} \right) \geq 0, \end{aligned}$$

$$\text{because } n_0 \geq 1 + \frac{3M_2}{\sqrt{M_3}}.$$

(e) Since $\Phi(K)$ is compact, we can choose $0 < R \leq (2n_0 M_6)^{-1}$ so that $\{t: |t - w| \leq R\}$ is contained in $\mathcal{A} \setminus \{0\}$. Further, we define:

$$D_R(z) := \{t: t = w + Re^{i\psi}, \psi \in [0, 2\pi]\},$$

$$L(z) := g(z, D_R(z)) = \{u: u = g(z, t), t \in D_R(z)\}, \text{ and}$$

$$r(z) := \text{dist}(L(z), 0) = \min\{|u|: u \in L(z)\}.$$

Since r is continuous on K and positive, there is an r_0 greater than 0 with $r_0 = \min\{r(z): z \in K\}$. So the set $\{u: |u| < r_0\}$ lies in the interior of $L(z)$ for all z in K . Therefore, we can deduce that for all z in K there exists a $\delta(z) > 0$ satisfying $|g(z, we^{i\delta(z)})| = r_0$, $|w(e^{i\delta(z)} - 1)| \leq R$, and $|g(z, we^{i\psi})| < r_0$ for all $\psi \in [0, \delta(z))$. Finally, we set $\delta_0 > 0$ with $\delta(z) \geq \delta_0$ for all z in K .

(f) Now, we define $k(z) := we^{i\delta_0}$ and $\kappa(z) := g(z, k(z))^2$, so we see that $\kappa(z)$ is continuous on K and $\Re(\kappa(z))$ is greater than 0 for all z in K . Considering the real part condition, there must be a κ_0 and a κ_{\max} with:

$$\kappa_0 > 0, \quad \kappa_0 = \min\{\Re(\kappa(z)): z \in K\}, \quad (3.22)$$

$$\kappa_{\max} > 0, \quad \kappa_{\max} = \max\{|\kappa(z)|: z \in K\}. \quad (3.23)$$

(g) For z in K we get (see [7, p. 123, set $n = 2$ and $z = n$]),

$$\int_w^{\kappa(z)} e^{-np(z, t)} q(t) dt = e^{-np(z, w)} \int_0^{\kappa(z)} e^{-nv} f(v) dv,$$

with $v = u^2 = (g(z, t))^2$, $f(v) = q(t) \frac{dt}{dv} = q(t) / \frac{\partial p(z, t)}{\partial t}$ and

$$f(v) = \sum_{s=0}^1 a_s v^{(s-1)/2} + \sqrt{v} f_2(v), \quad f_2(v) = \mathcal{O}(1), \quad v \rightarrow 0.$$

So we deduce:

$$\int_0^{\kappa(z)} e^{-nv} f(v) dv = \sum_{s=0}^1 \Gamma\left(\frac{s+1}{2}\right) \frac{a_s}{n^{(s+1)/2}} - \varepsilon_{2,1}(n, z) + \varepsilon_{2,2}(n, z).$$

The error terms can be estimated uniformly:

(g1) As mentioned above, the coefficient a_1 does not have to be taken into account, so it follows that $\varepsilon_{2,1}(n, z) = \Gamma(\frac{1}{2}, \kappa(z)n) (q_0/\sqrt{2p_0}) 1/\sqrt{n}$. Further, the incomplete gamma function holds:

$$\begin{aligned} \Gamma(\tfrac{1}{2}, \kappa(z)n) &= \int_{\kappa(z)n}^{\infty} e^{-t} t^{-1/2} dt = e^{-\kappa(z)n} \int_{\kappa(z)n}^{\infty} e^{-(t-\kappa(z)n)} t^{-1/2} dt \\ &= e^{-\kappa(z)n} \int_0^{\infty} e^{-x} (x + \kappa(z)n)^{-1/2} dx. \end{aligned}$$

By (3.22) we obtain $|\Gamma(\frac{1}{2}, \kappa(z)n)| \leq e^{-\kappa_0 n} (\kappa_0)^{-1/2}$, and by (3.20), $|\varepsilon_{2,1}(n, z)| \leq e^{-\kappa_0 n} (\kappa_0)^{-1/2} M_1 (\frac{M_4}{2})^{1/2} 1/\sqrt{n}$; that means

$$\varepsilon_{2,1}(n, z) = \frac{1}{\sqrt{n}} \mathcal{O}(e^{-\kappa_0 n}) \quad (3.24)$$

and the \mathcal{O} -term holds uniformly for all z in K .

(g2) Because $f_2(v) = v^{-1/2} (f(v) - \sum_{s=0}^1 a_s v^{(s-1)/2}) = \sum_{s=2}^{\infty} a_s v^{(s/2)-1}$ and $f_2(v) = \mathcal{O}(1)$, $v \rightarrow 0$, there must be an M_7 greater than 0 with $|f_2(v)| \leq M_7$ for all z in K , $|v|$ less than r_0^2 , and it follows that

$$\begin{aligned} |\varepsilon_{2,2}(n, z)| &= \left| \int_0^{\kappa(z)} e^{-nv} v^{1/2} f_2(v) dv \right| \\ &= \left| \kappa(z) \frac{1}{n^{3/2}} \int_0^n e^{-\kappa(z)x} (\kappa(z))^{1/2} x^{1/2} f_2\left(\frac{\kappa(z)x}{n}\right) dx \right| \\ &\leq (\kappa_{\max})^{3/2} n^{-3/2} \int_0^{\infty} e^{-\kappa_0 x} x^{1/2} M_7 dx =: M_8 n^{-3/2}; \end{aligned}$$

that means

$$\varepsilon_{2,2}(n, z) = \mathcal{O}(n^{-3/2}), \quad (3.25)$$

and the \mathcal{O} -term holds uniformly for all z in K .

(g3) The final error to calculate is the value of the integral along the arc of the semicircle from $we^{i\delta_0}$ to $-w$. The function $\tilde{g}(z, \psi) := \Re(p(z, we^{i\psi}) - p(z, w))$ is continuous on $K \times [\delta_0, \pi]$ and we deduce from the real part condition that there is an M_9 greater than 0 with $\tilde{g}(z, \psi)$ not less than M_9 for all (z, ψ) in $K \times [\delta_0, \pi]$. So we get

$$\begin{aligned} \left| \int_{k(z)}^{-w} e^{-np(z, t)} q(t) dt \right| &= \left| e^{-np(z, w)} \int_{k(z)}^{-w} e^{-n(p(z, t) - p(z, w))} q(t) dt \right| \\ &\leq |e^{-np(z, w)}| e^{-nM_9} M_1 \left| \int_{k(z)}^{-w} 1 dt \right| \\ &\leq |e^{-np(z, w)}| e^{-nM_9} M_1 M_{10}, \end{aligned}$$

with M_{10} greater than 0. Altogether, we obtain

$$\int_{k(z)}^{-w} e^{-np(z, t)} q(t) dt = e^{-np(z, w)} \mathcal{O}(e^{-nM_9}), \quad (3.26)$$

where the \mathcal{O} -term holds uniformly for all z in K again.

(h) From Theorem 3.2 and (3.24), (3.25), (3.26), and (3.16) we finally obtain:

$$\begin{aligned} Q_n(z) &= G_n(z)(1 + \mathcal{O}(e^{-\kappa_0 n}) + \mathcal{O}(n^{-1}) + \sqrt{n} \mathcal{O}(e^{-nM_9})) \\ &= G_n(z)(1 + \mathcal{O}(n^{-1})). \end{aligned}$$

The \mathcal{O} -term holds uniformly for all z in K and so the proof is completed. ■

3.3. The Airy-asymptotics

Finally, we give an Airy-asymptotics for Q_n as $z \rightarrow -e$:

THEOREM 3.4. *Let $z_n = -e(1 - (6n^2)^{-1/3} s)$, $s \in \mathbb{C}$, then, as $n \rightarrow \infty$,*

$$\begin{aligned} Q_n(z_n) &= \frac{n!}{\pi} (-1)^n \exp \left\{ (e-1) \left(n - \left(\frac{n}{6} \right)^{1/3} s \right) \right\} \left(\frac{6}{n} \right)^{1/3} \\ &\quad \times (A(s) + \mathcal{O}(n^{-1/3})), \end{aligned}$$

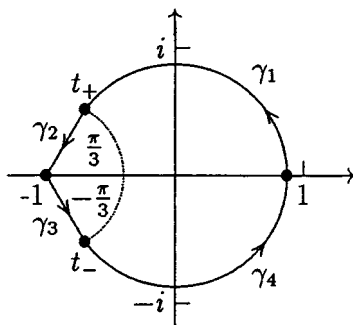


FIG. 2. The path of integration for the Airy-asymptotics.

where $A(s)$ is Airy's function. $A(s)$ is an entire function given by

$$A(s) = \frac{1}{2\pi i} \int_L \exp \left\{ \frac{1}{3} t^3 - st \right\} dt, \quad s \in \mathbb{C},$$

where L is any contour which begins at infinity in the sector $-\frac{\pi}{2} < \text{ph}(t) < -\frac{\pi}{6}$ and ends at infinity in the sector $\frac{\pi}{6} < \text{ph}(t) < \frac{\pi}{2}$; see [9, p. 90] and [8, p. 377]. Further, the \mathcal{O} -term holds uniformly for s in K , K compact.

Proof. The result can be proved in a similar way as in [8, pp. 232–235] for Laguerre polynomials. The main difference is the path of integration γ_0 , which should be chosen here in the following way (see Fig. 2):

Let θ be in $(0, \frac{1}{12})$, $t_+ = t_+(n) := -1 + (6n^{-1})^{1/3} n^\theta e^{\pi i/3}$, $\alpha_+ := \text{ph}(t_+) \in (0, \pi)$, $t_- = t_-(n) := -1 + (6n^{-1})^{1/3} n^\theta e^{-\pi i/3}$, $\alpha_- := \text{ph}(t_-) \in (\pi, 2\pi)$ and $r_n := |t_+|$. Further we define

$$\gamma_0 := \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4,$$

$$\gamma_1 := \{t: |t| = r_n, \text{ph}(t) \in [0, \alpha_+]\},$$

$$\gamma_2 := \{t: t = -1 - (6n^{-1})^{1/3} e^{\pi i/3} p, p \in [-n^\theta, 0]\},$$

$$\gamma_3 := \{t: t = -1 + (6n^{-1})^{1/3} e^{-\pi i/3} p, p \in [0, n^\theta]\}, \text{ and}$$

$$\gamma_4 := \{t: |t| = r_n, \text{ph}(t) \in [\alpha_-, 2\pi]\}.$$

The rest of the proof is quite similar to that mentioned above with the only difference that the calculations for γ_2 and γ_3 lead to the Airy function. ■

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